THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH4030 Differential Geometry
26 September, 2024 Tutorial

1. (Exercise 2-3.1 of [Car16]) Let $\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ be the unit sphere and let $A : \mathbb{S}^2 \to \mathbb{S}^2$ be the antipodal map $A(x, y, z) = (-x, -y, -z)$. Show that A is a diffeomorphism.

2. (Exercise 2-4.1 and 2-4.3 of [Car16])

   (a) Show that the equation of the tangent plane at (x_0, y_0, z_0) of a regular surface given by $f(x, y, z) = 0$, where 0 is a regular value of f, is

   $$f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) = 0.$$

   (b) Show that the equation of the tangent plane of a surface which is the graph of a differentiable function $z = f(x, y)$, at the point $p_0 = (x_0, y_0)$ is given by

   $$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

   Recall the definition of the differential df of a function $f : \mathbb{R}^2 \to \mathbb{R}$ and show that the tangent plane is the graph of the differential df_p.

3. (Exercise 2-4.11 of [Car16]) Show that the normals to a surface of revolution S given by

   $$X(u, v) = (f(u) \cos v, f(u) \sin v, g(u)), \quad f(u) \neq 0, g' \neq 0,$$

   all pass through the z axis.

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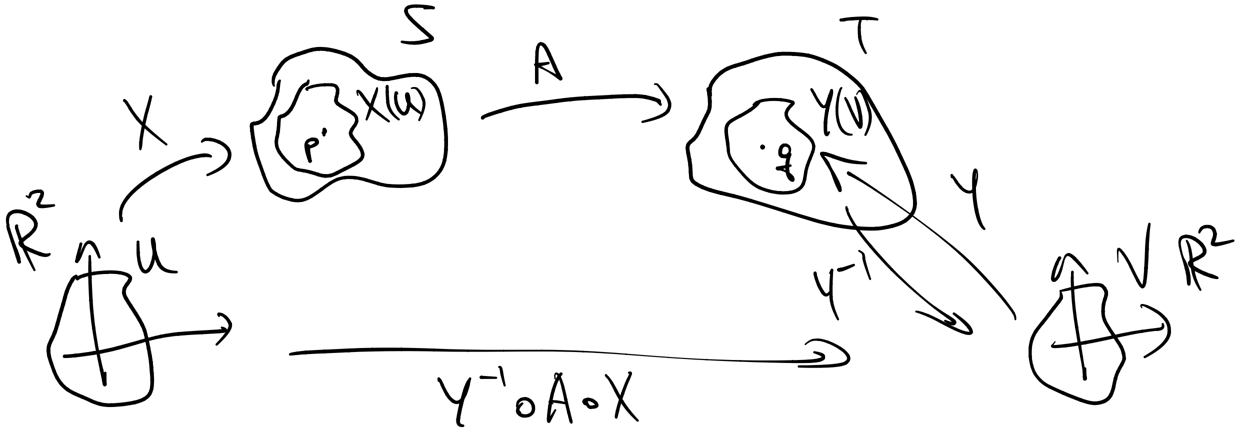
1. (Exercise 2-3.1 of [Car16]) Let $\mathbb{S}^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1\}$ be the unit sphere and let $A : \mathbb{S}^2 \to \mathbb{S}^2$ be the antipodal map $A(x,y,z) = (-x,-y,-z)$. Show that A is a diffeomorphism.

Pf: Need to check: • A is a smooth map between surfaces
• A admits a smooth inverse.

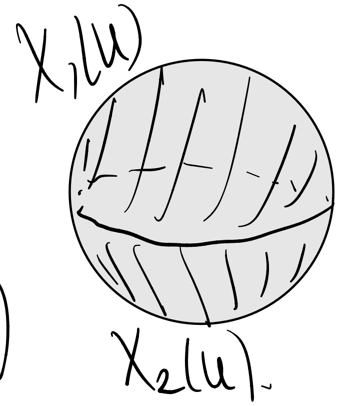


Need to find param. $X : U \subseteq \mathbb{R}^2 \to \mathbb{S}^2$ s.t. $p = X(u_0, v_0)$. $(u_0,v_0) \in U$ and $Y : V \subseteq \mathbb{R}^2 \to \mathbb{S}^2$ s.t. $q = A(p) \in Y(V)$ s.t. $Y^{-1} \circ A \circ X : U \to V$ is differentiable at (u_0, v_0).

$U = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 < 1\}$. and

$X_1 : U \to \mathbb{S}^2$ by $X_1(u,v) = (u, v, \sqrt{1-u^2-v^2})$

$X_2 : U \to \mathbb{S}^2$ by $X_2(u,v) = (-u, -v, -\sqrt{1-u^2-v^2})$



Sps $p \in X_1(U)$, then $p = X(u_0, v_0)$ and we have

$$A(p) = A(X_1(u_0,v_0)) = A(u_0, v_0, \sqrt{1-u_0^2-v_0^2}) = (-u_0, -v_0, -\sqrt{1-(-u_0)^2-(-v_0)^2})$$

So $A(p)$ lies in $X_2(U)$.

$$(X_2^{-1} \circ A \circ X_1)(u_0,v_0) = X_2^{-1}(-u_0, -v_0, -\sqrt{1-(-u_0)^2-(-v_0)^2}) = (-u_0, -v_0)$$

which is clearly differentiable as a map from $\mathbb{R}^2 \to \mathbb{R}^2$.

Similarly, if $p \in X_2(U)$, then $A(p) \in X_1(U)$.

and $X_1^{-1} \circ A \circ X_2$ is smooth at p.

Can repeat w/ other hemispheres.

$\Rightarrow$ A is smooth as a map from $S^2 \to S^2$.

Note that A is its own inverse.

2. (Exercise 2-4.1 and 2-4.3 of [Car16])

   (a) Show that the equation of the tangent plane at (x_0, y_0, z_0) of a regular surface given by $f(x, y, z) = 0$, where 0 is a regular value of f, is

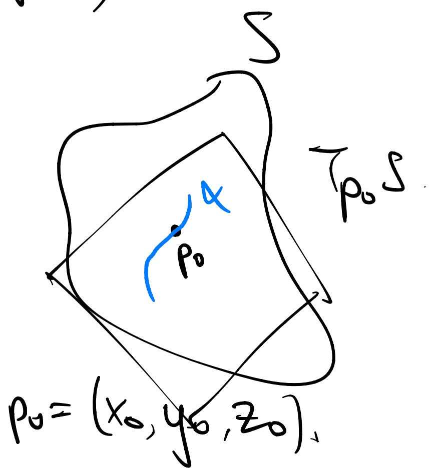
   $$f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) = 0.$$

   (b) Show that the equation of the tangent plane of a surface which is the graph of a differentiable function $z = f(x, y)$, at the point $p_0 = (x_0, y_0)$ is given by

   $$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

   Recall the definition of the differential df of a function $f : \mathbb{R}^2 \to \mathbb{R}$ and show that the tangent plane is the graph of the differential df_p.

Pf: a) We have $S = f^{-1}(0) = \{(x,y,z) \in \mathbb{R}^3 : f(x,y,z) = 0\}$.



Consider $w \in T_{p_0}S$.

Then there exist $\alpha : I \to S$ s.t. $\alpha(0) = p_0$, $\alpha'(0) = w$.

In particular, we have that $(f \circ \alpha)(t) = 0$ for all $t \in I$. $\alpha(t) = (x(t), y(t), z(t))$.

Recall derivative of f at p satisfies.

$$\langle df_p, w \rangle = \frac{d}{dt}\Big|_{t=0} (f \circ \alpha)(t) = (f \circ \alpha)'(0) = 0.$$

In particular, let $p = (x, y, z)$ and take $w = p - p_0 = (x - x_0, y - y_0, z - z_0)$,

then we got

$$f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) = 0.$$

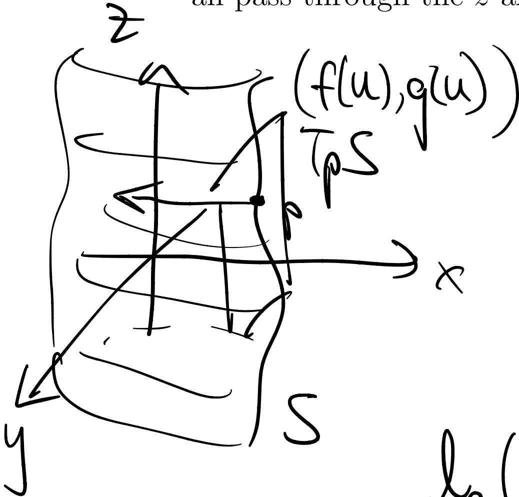
$/\!/$

b) Set $h : \mathbb{R}^3 \to \mathbb{R}$ by $h(x,y,z) = f(x,y) - z$

Then $S = h^{-1}(0)$ and 0 is a regular value of h. (check!).

and use part (a).

3. (Exercise 2-4.11 of [Car16]) Show that the normals to a surface of revolution S given by
$$X(u,v) = (f(u)\cos v, f(u)\sin v, g(u)), \quad f(u) \neq 0, g' \neq 0,$$
all pass through the z axis.



At each point $p = X(u,v)$ the normal line passing through p "line normal to surface S at p" is given by

$$l_p(\lambda) = X(u,v) + \lambda N(u,v)$$

where N is the normal vector to T_pS.

$$T_pS = \text{span}\{X_u, X_v\}, \quad \text{so } N = \frac{X_u \times X_v}{\|X_u \times X_v\|}.$$

$$X_u = (f'(u)\cos v, f'(u)\sin v, g'(u))$$
$$X_v = (-f(u)\sin v, f(u)\cos v, 0)$$

$$X_u \times X_v = (-f(u)g'(u)\cos v, -f(u)g'(u)\sin v, -f(u)f'(u))$$

$$l_p(\lambda) = (f(u)\cos v(1-\lambda g'(u)), f(u)\sin v(1-\lambda g'(u)), g(u) - \lambda f(u)f'(u))$$

So taking $\lambda = \frac{1}{g'(u)}$, $g' \neq 0$, then

$$l_p\left(\frac{1}{g'(u)}\right) = \left(0, 0, g(u) - \frac{f(u)f'(u)}{g'(u)}\right)$$ which lies on z-axis.